

Remarks on Non-Abelian Duality in $N=1$ Supersymmetric Gauge Theories

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Recently Seiberg has conjectured a duality symmetry connecting different theories of the supersymmetric QCD type. We provide support for this conjecture by analyzing a flat direction of the theory along which the two dual theories go over to the same theory in the IR.

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1. Introduction

The computation of exact non-perturbative results in quantum field theory is in general a very difficult problem. Nevertheless, in the context of supersymmetric field theories, there has recently been a substantial advance in our ability to perform such calculations. This advance was based on two main ideas – holomorphicity and strong-weak coupling duality.

Strong-weak coupling duality between electric and magnetic variables was originally proposed [1] as a symmetry of certain field theories. It is now believed that it may in fact have its origin in string theory [2]. This duality (usually called S-duality) is expected to be an exact symmetry of scale invariant supersymmetric theories, such as the $N = 4$ supersymmetric gauge theory and $N = 2$ supersymmetric gauge theories with certain matter contents [3],[4]. However, it may also be used in asymptotically free theories which have a dynamically generated scale Λ , such as some $N = 2$ [4-6] and $N = 1$ [7] gauge theories. It enables us to obtain exact non-perturbative information about the Coulomb phase of these theories, in which the symmetry is broken to an abelian subgroup. The understanding of these duality transformations in the case of an unbroken non-abelian gauge symmetry is still far from complete.

The description of supersymmetric field theories includes several holomorphic objects, whose holomorphicity may be used (together with the symmetries of the theory and various limits in which the behavior of the theory is known) to compute them exactly (see [8] for a recent review on this subject). In particular one may obtain by this method the exact low energy effective lagrangian describing $SU(N_c)$ supersymmetric QCD (SQCD) [9-15] in terms of gauge-invariant variables, when the number of quark flavors N_f is at most $N_c + 1$. In these cases the theory was found to be in a confining phase at the origin of moduli space (where the gauge symmetry is unbroken). However, for $N_f > N_c + 1$ it was not possible to find a consistent IR description of the theory in terms of gauge-invariant variables.

In a recent paper [16], Seiberg has conjectured that for $N_f > N_c + 1$, the theory of SQCD with gauge group $SU(N_c)$ and N_f quark flavors is equivalent to a theory of SQCD with gauge group $SU(N_f - N_c)$, whose matter content includes N_f quark flavors and additional gauge-singlet fields. For $N_f \geq 3N_c$ the original theory is not asymptotically free any more, and is, therefore, free in the IR, while for $N_c + 2 \leq N_f \leq \frac{3}{2}N_c$ the dual theory is IR free, so that it can be regarded as an appropriate description of the original theory in the IR – a non-abelian Coulomb phase of QCD. In the intermediate range $\frac{3}{2}N_c < N_f < 3N_c$

Seiberg conjectured that both theories have an IR fixed point, so that the appropriate IR description of the theory is in terms of some superconformal field theory. In the case of an $SO(N)$ gauge group, also discussed in [16], this sort of duality between theories was shown to be related to the electric–magnetic duality discussed above. This was possible since in this case the symmetry may be broken to an abelian subgroup ($SO(2)$) so that there exist (semi–classically) magnetic monopoles and the electric–magnetic duality transformation is relatively well understood. However, for an $SU(N_c)$ gauge group with matter in the fundamental representation, the gauge group cannot be broken (at least semi–classically) into an abelian subgroup, so that there are (semi–classically) no magnetic monopoles, and the connection between the duality of [16] and the electric–magnetic duality is, therefore, still unclear.

The evidence given in [16] for the duality conjecture was mostly kinematical in nature. Seiberg showed that the ’t Hooft anomaly conditions for equating the two theories are satisfied, and that they have the same flat directions and gauge–invariant observables. The dynamical evidence for the duality conjecture relied mainly on the possibility of flowing between theories with different N_f by giving some quarks a large mass and integrating them out. Seiberg showed that the duality conjecture is consistent with this flow, and that by decreasing the number of flavors to $N_c + 1$ or less one regains the known descriptions of SQCD for that number of flavors, in terms of the quarks or in terms of the gauge–invariant mesons and baryons. In this letter we wish to provide more dynamical support for the duality conjecture, by analyzing a flat direction of the two theories along which both theories become weakly coupled in the IR, enabling us to directly compare them. This was not possible in the cases analyzed in [16] for which whenever one theory was weakly coupled the other one was strongly coupled, so that no direct comparison could be made. In all the cases we will analyze we will find, in fact, that both theories go over to the same effective field theory in the IR, so that obviously they are equivalent there. We consider this to be important supporting evidence for the duality conjecture.

We begin in section 2 by reviewing the relevant parts of [16] which define the duality transformation. In section 3 we analyze a particular flat direction of both theories, involving a non–zero vacuum expectation value of a baryon field. We find that along this flat direction both theories become free in the IR with the same massless particle content. In section 4 we add perturbations to the original theories, and find that after adding them we get, along the flat direction we are analyzing, the same interacting IR field theory from both theories. We end in section 5 with a summary of our results and some speculations.

2. Flat directions in the dual theories

The duality transformation described in [16] connects two different theories of the supersymmetric QCD type, with generically different gauge groups. Both theories have enough matter fields in the fundamental representation so that they are not in a confining phase at the origin of moduli space. The first theory has only matter fields in the fundamental representation and no superpotential (following [16] we will call this theory the “electric” theory), while the second theory has additional gauge singlet fields and a non-trivial superpotential (we will call this the “magnetic” theory). Here we will only analyze the case of an $SU(N)$ gauge group.

The “electric” theory in this case is an $SU(N_c)$ gauge theory with N_f flavors of quarks, Q_a^i in the N_c representation and \tilde{Q}_i^a in the \overline{N}_c representation ($i, \tilde{i} = 1, \dots, N_f; a = 1, \dots, N_c$). The anomaly free global symmetry of this theory is (for $N_c > 2$)

$$SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R \quad (2.1)$$

with the quarks transforming as

$$\begin{aligned} Q & \quad (N_f, 1, 1, \frac{N_f - N_c}{N_f}) \\ \tilde{Q} & \quad (1, \overline{N}_f, -1, \frac{N_f - N_c}{N_f}). \end{aligned} \quad (2.2)$$

The interesting gauge invariant operators which characterize the moduli space and which we will study are

$$\begin{aligned} M_i^i &= Q^i \tilde{Q}_i \\ B^{[i_1, \dots, i_{N_c}]} &= Q^{i_1} \dots Q^{i_{N_c}} \\ \tilde{B}_{[\tilde{i}_1, \dots, \tilde{i}_{N_c}]} &= \tilde{Q}_{\tilde{i}_1} \dots \tilde{Q}_{\tilde{i}_{N_c}} \end{aligned} \quad (2.3)$$

where a summation over color indices in the first definition and an anti-symmetrization over color indices in the other two definitions are implied.

For $N_f < N_c + 2$ this theory confines, and has an infra-red description in terms of gauge invariant fields (summarized in [15]). For $N_f \geq N_c + 2$, the only case we will be interested in here, the quantum moduli space is the same as the classical one [15], as can be shown by turning on a tree level mass term $W_{tree} = \text{Tr} (mM)$ and finding

$$\langle M_i^i \rangle = \Lambda^{\frac{3N_c - N_f}{N_c}} (\det m)^{\frac{1}{N_c}} \left(\frac{1}{m} \right)_{\tilde{i}}^i. \quad (2.4)$$

Then, by studying various limits of $m \rightarrow 0$, all the classically allowed values of M with $B = \tilde{B} = 0$ can be obtained. Presumably, by adding other perturbations it can be shown [16] that the classically allowed values with non-zero B and \tilde{B} are also in the quantum moduli space. Thus, there is no quantum superpotential, and the flat directions of this theory in which the D terms are zero, are (up to gauge and global rotations) of the form

$$Q = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{N_c} \end{pmatrix}; \quad \tilde{Q} = \begin{pmatrix} \tilde{a}_1 & & & \\ & \tilde{a}_2 & & \\ & & \ddots & \\ & & & \tilde{a}_{N_c} \end{pmatrix} \quad (2.5)$$

with

$$|a_i|^2 - |\tilde{a}_i|^2 = \text{independent of } i. \quad (2.6)$$

The gauge invariant description of this moduli space is given in terms of the observables M , B and \tilde{B} [15]. Up to global symmetry transformations they are given by

$$M = \begin{pmatrix} a_1 \tilde{a}_1 & & & \\ & a_2 \tilde{a}_2 & & \\ & & \ddots & \\ & & & a_{N_c} \tilde{a}_{N_c} \end{pmatrix} \quad (2.7)$$

$$B^{1,\dots,N_c} = a_1 a_2 \dots a_{N_c}$$

$$\tilde{B}_{1,\dots,N_c} = \tilde{a}_1 \tilde{a}_2 \dots \tilde{a}_{N_c}$$

with all other components of M , B and \tilde{B} vanishing. Obviously the rank of M is at most N_c . If it is less than N_c , either $B = 0$ with \tilde{B} having at most rank one or $\tilde{B} = 0$ with B having at most rank one. If the rank of M is equal to N_c , both B and \tilde{B} have rank one and the product of their non-zero eigenvalues is the same as the product of the non-zero eigenvalues of M .

The conjectured dual to this theory, the “magnetic” theory, is an $SU(N_f - N_c)$ gauge theory, with N_f flavors of quarks and additional independent singlet fields corresponding to the mesons of the “electric” theory. We will denote the quarks of this theory by q_i and \tilde{q}^i : q_i transforms as $N_f - N_c$ of the color group and \tilde{q}^i transforms as $\overline{N_f - N_c}$. The global symmetry group of this theory is the same as that of the “electric” theory, given by (2.1). The quantum numbers of the quarks in this theory may be determined so that the baryons

constructed from the “magnetic” quarks have the same quantum numbers as the baryons constructed from the “electric” quarks, enabling us to identify them. This leads to

$$\begin{aligned}
q & \quad \text{in} \quad (\bar{N}_f, 1, \frac{N_c}{N_f - N_c}, \frac{N_c}{N_f}) \\
\tilde{q} & \quad \text{in} \quad (1, N_f, -\frac{N_c}{N_f - N_c}, \frac{N_c}{N_f}) \\
M & \quad \text{in} \quad (N_f, \bar{N}_f, 0, 2(\frac{N_f - N_c}{N_f})).
\end{aligned} \tag{2.8}$$

It is easy to check [16] that this assignment of quantum numbers is anomaly free, and moreover it satisfies the ’t Hooft anomaly matching conditions. Thus, the global anomalies of this theory are exactly the same as those of the “electric” theory.

As it stands, the “magnetic” theory has an additional gauge-invariant field which was not present in the “electric” theory – the “magnetic meson” field $N_i^{\tilde{q}} = q_i \tilde{q}^i$. To identify the two theories we must get rid of this field, and this may be done by adding to the action a superpotential

$$W = M_i^{\tilde{q}} q_i \tilde{q}^i \tag{2.9}$$

(where the color indices are summed). After adding this term the “magnetic meson” operator is redundant – its coefficient can be absorbed in a shift of M . As we will see, this term is necessary for equating the flat directions of the two theories, and, as described in [16], it also enables us to return to the original theory when performing the duality transformation twice.

In the “electric” theory the global symmetry was enhanced to $SU(2N_f) \times U(1)_R$ when $N_c = 2$, since in that case the quarks and anti-quarks are in the same representation of the color group. In the “magnetic” theory we should, therefore, have an enlarged symmetry group in this case as well. It should, in particular, relate the mesons and baryons of this theory, so that its action on the “magnetic” quarks will be non-trivial. It is not clear how this enlarged symmetry of the “magnetic” theory may be seen directly, but we will assume its existence (as part of the duality conjecture). There is no enhanced symmetry in the “magnetic” theory when $N_f - N_c = 2$, since it is broken explicitly by the presence of the mesons and by the superpotential.

In the “electric” theory we saw that no quantum superpotential was generated, since all classically allowed values for the meson and baryon fields could be obtained in the full quantum theory by taking various limits of massive theories. In the “magnetic” theory

this is no longer true. This can be seen, for instance, by analyzing the flat direction in the classical moduli space in which M has a generic VEV while the squarks have zero VEV. Then, all the dual quarks are massive and the low energy “magnetic” gauge group leads to gluino condensation. Working out the M dependence of the low energy gauge theory, one can find [16] that a superpotential proportional to

$$\frac{(\det M)^{\frac{1}{N_f - N_c}}}{\Lambda^{\frac{3N_c - N_f}{N_f - N_c}}} \quad (2.10)$$

is generated. (We use the conventions of [16] for the dimensions of the various fields, which arrange the powers of the QCD scale Λ according to the dimensions of the fields at the UV fixed point of the “electric” theory, and add additional powers of Λ when looking at other limits in which the fields have different dimensions). By adding to the action a small mass term, $\text{Tr}(mM)$, and requiring that we get the expectation values (2.4), we find that the quantum effective superpotential of the “magnetic” theory along this flat direction is in fact

$$W_{eff} = M_{ij}^i q_i \tilde{q}^j - (N_f - N_c) \frac{(\det M)^{\frac{1}{N_f - N_c}}}{\Lambda^{\frac{3N_c - N_f}{N_f - N_c}}}. \quad (2.11)$$

We will assume that this is the exact superpotential along the flat directions for which the gauge group is confined, or equivalently completely Higgsed (see [7] for a discussion of how different effective superpotentials can arise in different phases of the theory). These are the flat directions we will be interested in here. The most general superpotential respecting the symmetries looks like the first term in (2.11) times a general function of the ratio of the two terms, but the choice above seems to be the only one for which the relevant flat directions in the “magnetic” theory are indeed equivalent to those of the “electric” theory. It is interesting to note that the additional quantum superpotential becomes irrelevant in the IR exactly when $N_f < \frac{3}{2}N_c$ (since the power of the meson superfield appearing is $\frac{N_f}{N_f - N_c}$ which is larger than 3 in this case). Thus, in this case the “magnetic” theory indeed becomes free in the IR.

The flat directions of the “magnetic” theory may now be determined by requiring that the D terms and the F terms coming from W_{eff} all vanish. It can easily be seen that if either q or \tilde{q} is zero, an expectation value of M of rank less than N_c is a flat direction of this theory. The D term in this case forces q (or \tilde{q}) to have $N_f - N_c$ equal eigenvalues. If both q and \tilde{q} are non-zero, the D and F terms force both of them to be of rank $N_f - N_c$ (so that B and \tilde{B} are non-zero), with the matrix $\tilde{q}q$ having $N_f - N_c$ equal non-zero eigenvalues.

M then has to be of rank N_c , with the product of the eigenvalues of M equaling (up to an appropriate power of Λ) the product of the non-zero elements of B and \tilde{B} .

We have thus found exactly the same flat directions in both theories when expressed in terms of the gauge-invariant variables. In both theories there is one flat direction in which the meson M has a rank less than N_c and B and \tilde{B} are both zero or at most one of them has one non-zero eigenvalue. Another flat direction is the one in which B and \tilde{B} both have one non-zero eigenvalue, in which case M is forced to be of rank N_c with the product of its non-zero eigenvalues equal to the product of the non-zero eigenvalues of B and \tilde{B} .

3. The baryonic flat direction

Most of the flat directions described above were analyzed in [16] for both theories, and it was shown that by going along them and integrating out the fields that become massive, the IR description of the two theories becomes that of another pair of dual theories. An interesting flat direction which was not analyzed in [16] is the direction in which the baryon B gets a VEV while the anti-baryon \tilde{B} does not. In this case, which we will call the baryonic flat direction, we will show that the two theories in fact go over to exactly the same theory in the IR. We consider this to be important support for the conjecture that the two theories are in fact the same in the IR.

Let us begin with the simplest case, in which the meson expectation value is zero. In the “electric” theory this corresponds to squark expectation values of the form (up to gauge and global rotations)

$$Q = \begin{pmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{pmatrix}; \quad \tilde{Q} = 0 \quad (3.1)$$

and in the “magnetic” theory this corresponds to

$$q = \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix}; \quad \tilde{q} = 0; \quad M = 0 \quad (3.2)$$

where the baryon VEV in the “electric” theory equals a^{N_c} , and the baryon VEV in the “magnetic” theory (related to the “electric” VEV by appropriate powers of Λ) equals $A^{N_f - N_c}$. In both cases the gauge symmetry is completely broken, so that both theories are in a Higgs phase. This is unlike the cases analyzed in [16] for which one theory was in a Higgs phase when the other was in a confining phase, although the two phases are of course indistinguishable in this case since we have matter only in the fundamental representation of the gauge group. Note that the baryon composed of the first N_c quarks in the “electric” theory is equivalent to the baryon composed of the last $N_f - N_c$ quarks in the “magnetic” theory.

In the “electric” theory, the $SU(N_f)$ flavor symmetry acting on the quarks Q is broken to $SU(N_c) \times SU(N_f - N_c) \times U(1)_F$. The two original $U(1)$ symmetries are explicitly broken by the VEV, but a combination of each one of them with the $U(1)_F$ from the flavor symmetry remains unbroken. A diagonal sub-group of the product of the $SU(N_c)$ color symmetry and the $SU(N_c)$ coming from the flavor group also remains unbroken, so that the total global symmetry of the theory is

$$SU(N_c) \times SU(N_f - N_c) \times SU(N_f) \times U(1)_{\hat{B}} \times U(1)_{\hat{R}}. \quad (3.3)$$

Of the first N_c quark flavors (each one consisting of N_c chiral superfields), $N_c^2 - 1$ chiral superfields join the gauge bosons by the Super-Higgs mechanism to become $N_c^2 - 1$ massive vector multiplets of mass $g_e |a|$ (where g_e is the coupling constant of the “electric” theory). The remaining combination is a massless singlet labeling the flat direction which will not play a role in the subsequent discussions. The other quarks and anti-quarks all remain massless, and their quantum numbers under the new global symmetry can easily be found to be

$$\begin{aligned} Q & \quad (N_c, N_f - N_c, 1, \frac{N_f}{N_f - N_c}, 1) \\ \tilde{Q} & \quad (\bar{N}_c, 1, \bar{N}_f, -1, \frac{N_f - N_c}{N_f}). \end{aligned} \quad (3.4)$$

The massless Q quarks (and one real scalar from the massless singlet mentioned above) may be interpreted as Goldstone bosons for the breaking of the global flavor symmetry. Integrating out the massive fields leaves us in this case with a free theory with this field content.

The “magnetic” theory along this flat direction behaves in a similar fashion. The $SU(N_f)$ flavor symmetry acting on the quarks breaks here as well to $SU(N_f - N_c) \times$

$SU(N_c) \times U(1)_F$, with a diagonal subgroup of the $SU(N_f - N_c)$ color symmetry and the $SU(N_f - N_c)$ symmetry from the breaking of the flavor group remaining a global symmetry of the theory. As above, $(N_f - N_c)^2 - 1$ quarks join the gauge bosons to become massive vector superfields of mass $g_m|A|$, while another combination of the first $N_f - N_c$ quarks is massless and labels the flat direction. However, in this case the superpotential also gives a mass to some of the other fields. All the anti-quark fields \tilde{q} get a mass $|A|$, as do the mesons M_i^j for $i = N_c + 1, \dots, N_f$. The fields that remain massless are, thus, N_c flavors of quarks, and $N_c N_f$ meson fields. The global symmetries that remain are exactly the same as those we found in (3.3) for the “electric” theory, and again it is easy to compute the quantum numbers of the remaining massless fields, which are

$$\begin{aligned} q & \quad (\overline{N}_c, N_f - N_c, 1, \frac{N_f}{N_f - N_c}, 1) \\ M & \quad (N_c, 1, \overline{N}_f, -1, \frac{N_f - N_c}{N_f}). \end{aligned} \tag{3.5}$$

As in the “electric” theory, the massless fields q may be interpreted as Goldstone bosons for the global symmetry breaking.

Up to conjugation of the $SU(N_c)$ subgroup of the global symmetry group (which is obviously a matter of convention), these are exactly the same quantum numbers as those we found above for the massless fields Q, \tilde{Q} in the “electric” theory. Here too we can integrate out the massive fields, and we find that the remaining superpotential for the massless fields is zero, so that the theory is free in the IR. The two dual theories thus lead to exactly the same theory in the IR along the baryonic flat direction, providing support for the conjecture that the theories are indeed the same in the IR. Of course, this is a trivial case since both theories are free in the IR. Later we will see examples where the two dual theories go over to the same interacting field theory in the IR.

Once again, special notice should be given to the $N_c = 2$ case, in which the “electric” theory has a larger symmetry. Along the flat direction we are analyzing, the global symmetry in this case is $SU(N_c) \times SU(2N_f - N_c) \times U(1)_{\hat{R}}$ and all massless quarks sit in the same fundamental representation of $SU(2N_f - N_c)$. For the duality to hold, we need a similar symmetry connecting the massless fields q and M in the “magnetic” theory. We will argue that the symmetry we need can in fact be derived from the symmetry connecting the mesons and the baryons in the “magnetic” theory (which we assume exists in the IR). This is the case since when we replace the heavy quarks by their VEVs (when integrating them out), we find that the baryon containing $N_f - N_c - 1$ of the last $N_f - N_c$ (massive)

quark flavors, and one of the remaining N_c massless flavors, equals $A^{N_f - N_c - 1}$ times a component of the massless quark flavor. This component has its color index equal (up to a shift by N_c) to the flavor index missing (among the last $N_f - N_c$ flavor indices) from the definition of the baryon. Thus, the massless quarks in this case are equivalent to baryons (as found in [16] also for other cases) so that the additional symmetry required follows straightforwardly from the symmetry between the mesons and the baryons.

The above results can easily be generalized to the case in which the meson matrix also gets a VEV of rank k for $k < N_c$ (with the same baryon VEVs). For example, if we have a meson VEV with k non-zero eigenvalues which are all equal, the global symmetry breaks in both theories into

$$SU(k) \times SU(N_c - k) \times SU(N_f - k) \times SU(N_f - N_c) \times U(1) \times U(1)_B \times U(1)_R \quad (3.6)$$

and again we can find in the IR a free field theory with the same massless particle content originating from both theories.

An analysis similar to the one we did in this chapter may also be performed for the case of $N_f = N_c + 1$, whose exact superpotential was given in [15]. Indeed, if we identify the baryon field used there with the dual quark (up to an appropriate scale factor), the superpotential given in [15] is exactly the same as (2.11). Thus, the analysis for that case is exactly the same as the one we performed here, so that also there the two descriptions go over to the same theory in the IR along the baryonic flat direction.

4. Perturbations along the baryonic flat direction

In order to verify that the duality conjecture holds, one should in principle compute correlation functions in both theories and see that they give the same result. For example, since the global symmetry is the same in both theories, one can always try to compare correlation functions of global symmetry currents. Unfortunately, since the gauge coupling in one theory gets stronger when that of the other theory gets weaker, it seems impossible to perform such computations perturbatively in both theories as long as the gauge symmetry is unbroken.

However, we have seen that there is a flat direction along which the two theories both go into the same theory in the IR, so that along this flat direction we can directly compare the IR correlation functions of the two theories. These are, however, trivial in this case since both theories are free below the symmetry breaking scale. It would be more

interesting if we could find a phase in which both theories would go over to a non-trivial weakly coupled theory in the IR. Then we could compare non-trivial correlation functions between the two theories. We will do this by adding to the action perturbations by various gauge invariant superfields, which do not destroy the baryonic flat direction. We will find in all such cases that the resulting effective action in the IR is the same in both theories, so that obviously all correlation functions are the same. For simplicity we will only analyze in this section the case in which the meson VEV is zero.

The gauge invariant objects we can perturb the action by are essentially the meson and baryon operators. The simplest perturbation is obviously by a meson operator, and if we choose it to be a meson operator which does not include any of the first N_c quark flavors, it will not destroy the baryonic flat direction. Let us choose [16] a perturbation of the form $W = m M_{N_f}^{N_f}$. Note that the dimension of m in the UV of the “electric” and “magnetic” theories is not the same. We will denote the constant m in the “electric” theory by m_e , and in the “magnetic” theory by m_m , and we will find below what the relation between them must be.

In the “electric” theory, all this perturbation does is to give a mass m_e to the last quark and anti-quark flavors, which were so far massless. If m_e is much smaller than the symmetry breaking scale a , we can integrate out the same massive fields as above, and remain with the same field content as in (3.4) but with a mass m_e for one quark and one anti-quark flavor. In the “magnetic” theory, however, the addition of this term changes the F -term equation of the relevant meson (which is massive there) to $q_{N_f} \tilde{q}^{N_f} + m_m = 0$. Thus, if we wish to keep the same flat direction as before, we must have $\langle \tilde{q}_{N_f - N_c}^{N_f} \rangle = -\frac{m_m}{A}$. In fact $\langle q_{N_f}^{N_f - N_c} \rangle$ will in this case be corrected by terms of order $\frac{m_m^2}{A^3}$ to keep the D -terms zero, but we will assume $m_m \ll A^2$ and ignore these terms (the opposite case of large m_m , for which the fields of mass m_m are integrated out, was analyzed in [16]). Through the superpotential, this leads to a mass $|\frac{m_m}{A}|$ for the quarks $q_i^{N_f - N_c}$ of all flavors i , and for the mesons $M_{N_f}^i$. After integrating out the fields whose mass is of order A , we remain with a free theory, with a mass $|\frac{m_m}{A}|$ for the $(N_f - N_c)$ -th color component of the quarks which were massless before, and for the mesons $M_{N_f}^i$ for $i = 1, \dots, N_c$ which were massless before. These are exactly the particles that we identified above (in the discussion of the free theory) with the last flavor of quark and anti-quark in the “electric” theory. Thus, we find that (up to a factor of A which we means we should identify $m_m = A m_e$) the two dual theories again go over to the same theory in the IR. Again it is a free theory but this time with a non-trivial (and equal) mass spectrum. Analogous results can also be obtained by

analyzing more complicated mesonic perturbations, as long as they leave the baryonic flat direction flat.

The next perturbation one can think of is a perturbation by the baryon operators B and \tilde{B} . These terms are non-renormalizable in the “electric” theory if $N_c > 3$ and in the “magnetic” theory if $N_f - N_c > 3$. However, since we are only looking at the IR effective action, we can add them even in those cases (as done for instance in [17]), assuming that they can be derived from some renormalizable interaction at higher energies. Let us begin by analyzing the perturbation by one baryon operator, which is in the “electric” theory

$$W = b_e B^{i_1, \dots, i_{N_c}} = b_e \det(Q_a^i)_{i=1, \dots, i_{N_c}, a=1, \dots, N_c}. \quad (4.1)$$

This perturbation does not destroy the baryonic flat direction as long as at least two of the quark flavor indices involved are greater than N_c . We will analyze here the simplest case in which all flavors involved are greater than N_c and $N_f = 2N_c$. In this case all the quarks involved remain massless along the baryonic flat direction. We are left, after integrating over the massive fields, with the same superpotential (4.1) (which leads to a non-trivial interaction near the IR), which is simply the determinant of the matrix of massless quark fields. The analogous interaction in the “magnetic” theory is the same, with the baryon operator equal to the determinant of the first $N_c = N_f - N_c$ quark flavors. Once again all these quarks remain massless along the baryonic flat direction. We remain, after integration over the massive fields, with a superpotential which includes only the perturbation, which is again the determinant of the matrix of massless quarks. This is the first example in which we find that the two dual theories go over to the same *interacting* field theory in the IR limit. Other baryon perturbations, for general values of N_f , work in a similar fashion, leading in all cases to the same superpotential in both theories after integrating out the massive fields (which for this type of perturbation is generally equivalent to replacing the massive quarks by their VEVs).

A more interesting case is adding a perturbation by the anti-baryon operator \tilde{B} . The baryonic flat direction survives this type of perturbation for any value of N_f, N_c and for any choice of anti-quark flavors composing the anti-baryon. For definiteness let us take the perturbation in the “electric” theory to be

$$W_e = \tilde{b}_e \tilde{B} = \tilde{b}_e \det(\tilde{Q}_i^a)_{a, \tilde{i}=1, \dots, N_c} \quad (4.2)$$

and in the “magnetic” theory it is then

$$W_m = \tilde{b}_m \det(\tilde{q}_a^i)_{a=1,\dots,N_f-N_c, \tilde{i}=N_c+1,\dots,N_f}. \quad (4.3)$$

In the “electric” theory this perturbation is similar to the previous case. It does not affect the baryonic flat direction, and remains after integrating out the massive fields, since all the fields involved are massless. In the “magnetic” theory, however, the fields involved are all massive and one should carefully integrate out the massive fields using a superpotential that is the sum of W_{eff} from (2.11) and W_m . Performing this calculation leads to an effective potential of the form

$$W_{eff} = \tilde{b}_m \frac{\Lambda^{-(3N_c-N_f)}}{A^{N_f-N_c}} \det(M_i^{\tilde{i}})_{\tilde{i}, i=1,\dots,N_c}. \quad (4.4)$$

Up to scale factors arising from transforming the relevant operators to the “magnetic” theory, this is exactly the same interaction as we found in the “electric” theory, since the mesons involved were identified (in the IR where we have a free theory) with the anti-quarks involved in the “electric” interaction. This is another non-trivial example of obtaining the same IR theory when starting from the two dual theories.

In a similar fashion one can analyze more complicated perturbations, like powers of meson operators. In all cases that were checked one obtains the same interacting IR theory when starting from the two dual theories and integrating out the massive fields along the baryonic flat direction. Namely, the interaction in the “electric” theory in terms of the massless quarks and anti-quarks turns out to be the same as the interaction in the “magnetic” theory in terms of the massless quarks and mesons. For this to hold the “electric” and “magnetic” quarks and the “electric” anti-quarks and “magnetic” mesons are identified as in the free theory we analyzed in the previous section.

The analysis of this chapter may also be generalized to the case $N_f = N_c + 1$. There is no baryon perturbation in that case, since it always ruins the flat direction we are checking. However, the meson and anti-baryon perturbations work there exactly the same as they do here, giving once again the same interacting IR theory starting from the original two dual descriptions.

5. Summary and Conclusions

Establishing the duality conjecture of [16] in a convincing manner requires the comparison of physical quantities between the “electric” and “magnetic” theories. Such quantities could be, for instance, correlators of the global symmetry currents, or some critical exponents like the derivative of the beta function at the IR fixed point². Unfortunately this sort of comparison is hard to achieve, since usually at least one of the theories is strongly coupled. In this letter we provided support for the duality conjecture by studying a particular flat direction along which both theories become weakly coupled in the IR. We found that, along this flat direction, both theories give rise to the same effective theory in the IR. Hopefully our calculations, and perhaps similar calculations which may be performed for other gauge groups, will assist in shedding some light on the origin of this duality symmetry, and on its relation to the electric–magnetic duality, which are still not clear.

The duality symmetry in [16] is presented as an equality between the IR descriptions of two theories, and not as an exact equivalence between the two theories at all energy scales. It is an intriguing open question to compare the theories also in the UV. Naively, in the range $\frac{3}{2}N_c < N_f < 3N_c$ for which both theories are asymptotically free, we would expect the two theories to be different in the UV, as would be revealed (upon gauging part of the global symmetry) by experiments such as e^+e^- annihilation and deep inelastic scattering. In particular the value of R in the two theories is expected to be different since the quark content of the theories is different. It seems to us, however, that one should be more cautious in using arguments such as this. When the gauge coupling of the “magnetic” theory becomes weak in the UV, the superpotential becomes strong (since it is like a Φ^3 coupling in the WZ model when the gauge coupling can be neglected), so that it does not seem possible to actually perform any trustworthy perturbative calculations in the UV of the “magnetic” theory. Perhaps the superpotential only becomes strong at a scale much higher than Λ , in which case perturbative calculations would be trustworthy at energies between the two scales and would rule out the equivalence between the two theories. However, as long as the behavior of the coupling in the superpotential as a function of the scale is not well understood, it does not seem possible to rule out the possibility that the two theories will somehow conspire to give the same results in the UV as well. Clearly a better understanding of this issue is important in determining whether the theories could be the same in the UV or not.

² We would like to thank M. Peskin for discussions on this subject.

Another issue is that, assuming that for $\frac{3}{2}N_c < N_f < 3N_c$ the beta function has exactly one zero, the theory exists (at least formally) in one of three regimes. The gauge coupling is either exactly at the critical coupling, or it is below it and becomes weakly coupled in the UV, or it is above it and becomes strongly coupled in the UV. Since the duality transformation is supposed to interchange strong and weak coupling, it may very well be that the theory in the asymptotically free regime would be equivalent by duality to a theory in the non-asymptotically free regime. If this is true, a comparison between the theories when both are asymptotically free is not relevant. Note that both theories still must have the same behavior in the IR. A similar argument can of course be made when the beta function has more than one zero.

Obviously, much more work is needed in order to understand the origin and nature of this conjectured duality, and whether it holds only for the IR description of the theory or at higher energies as well.

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References

- [1] C. Montonen and D. Olive, Phys. Lett. **72B** (1977) 117; P. Goddard, J. Nuyts, and D. Olive, Nucl. Phys. **B125** (1977) 1
- [2] A. Sen, hep-th/9402002, Int. J. Mod. Phys. **A9** (1994) 3707; hep-th/9402032, Phys. Lett. **329B** (1994) 217
- [3] C. Vafa and E. Witten, hep-th/9408074, Nucl. Phys. **B431** (1994) 3
- [4] N. Seiberg and E. Witten, hep-th/9408099, Nucl. Phys. **B431** (1994) 484
- [5] N. Seiberg and E. Witten, hep-th/9407087, Nucl. Phys. **B426** (1994) 19
- [6] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, “Simple singularities and N=2 supersymmetric Yang–Mills theory”, hep-th/9411048, CERN-TH.7495/94, LMU-TPW 94/16; P. C. Argyres and A. E. Faraggi, “The vacuum structure and spectrum of N=2 supersymmetric SU(n) gauge theory”, hep-th/9411057, IASSNS-HEP-94-94
- [7] K. Intriligator and N. Seiberg, hep-th/9408155, Nucl. Phys. **B431** (1994) 551
- [8] N. Seiberg, “The Power of Holomorphy – Exact Results in 4D SUSY Field Theories”, to appear in the Proc. of PASCOS 94, hep-th/9408013, RU-94-64, IASSNS-HEP-94/57
- [9] G. Veneziano and S. Yankielowicz, Phys. Lett. **113B** (1982) 231; T.R. Taylor, G. Veneziano and S. Yankielowicz, Nucl. Phys. **B218** (1983) 493
- [10] I. Affleck, M. Dine, and N. Seiberg, Nucl. Phys. **B241** (1984) 493; Nucl. Phys. **B256** (1985) 557
- [11] V.A. Novikov, M.A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. **B223** (1983) 445; Nucl. Phys. **B229** (1983) 381; Nucl. Phys. **B260** (1985) 157
- [12] M.A. Shifman and A.I Vainshtein, Nucl. Phys. **B277** (1986) 456; Nucl. Phys. **B359** (1991) 571
- [13] D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, Phys. Rep. **162** (1988) 169 and references therein
- [14] N. Seiberg, hep-ph/9309335, Phys. Lett. **318B** (1993) 469
- [15] N. Seiberg, hep-th/9402044, Phys. Rev. **D49** (1994) 6857
- [16] N. Seiberg, “Electric–magnetic duality in supersymmetric non–abelian gauge theories”, hep-th/9411149, RU-94-82, IASSNS-HEP-94/98
- [17] K. Intriligator, N. Seiberg and S. Shenker, hep-ph/9410203, Phys. Lett. **342B** (1995) 152